

## UPDATE: REMARKS ON COUNTABLE TIGHTNESS

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ABSTRACT. The proof of Theorem 11 of the paper [5] relies on Lemma 10 of that paper. The offered proof of Lemma 10 had shortcomings, and I was recently asked for details. This note gives an alternative, complete proof of [5], Lemma 10.

In [5] we considered the selection principle  $S_1(\mathcal{A}, \mathcal{B})$  for specific instances of families  $\mathcal{A}$  and  $\mathcal{B}$  of sets. Recall that  $S_1(\mathcal{A}, \mathcal{B})$  denotes the statement that there is for each sequence  $(O_n : n \in \mathbb{N})$  of elements of  $\mathcal{A}$  a corresponding sequence  $(x_n : n \in \mathbb{N})$  such that for each  $n$  we have  $x_n \in O_n$ , and  $\{x_n : n \in \mathbb{N}\} \in \mathcal{B}$ .

There is a natural game, denoted  $G_1(\mathcal{A}, \mathcal{B})$  associated with this selection principle: This two-player game is played as follows. There is an inning for each  $n < \omega$ . In inning  $n$  player ONE selects and  $O_n \in \mathcal{A}$ , and then TWO responds by selecting an  $x_n \in O_n$ . A play  $(O_0, x_0, \dots, O_n, x_n \dots)$  is won by player TWO if the set  $\{x_n : n < \omega\}$  is an element of the family  $\mathcal{B}$ ; else, the play is won by player ONE.

If player ONE does not have a winning strategy in the game  $G_1(\mathcal{A}, \mathcal{B})$ , then the selection principle  $S_1(\mathcal{A}, \mathcal{B})$  holds of the pair  $\mathcal{A}, \mathcal{B}$ . For many topological families  $\mathcal{A}$  and  $\mathcal{B}$  it is the case that under appropriate circumstances also the converse holds: The selection principle  $S_1(\mathcal{A}, \mathcal{B})$  implies that ONE has no winning strategy in the game  $G_1(\mathcal{A}, \mathcal{B})$ . Theorem 11 of [5] was intended to demonstrate an extreme case of failure of this converse for well-studied examples of  $\mathcal{A}$  and  $\mathcal{B}$ .

More precisely: In [2] Sakai defined the notion of countable strong fan tightness at the point  $x$  of a topological space  $(X, \tau)$ . This notion is defined as follows: For the point  $x \in X$  we define

$$\Omega_x = \{A \subseteq X \setminus \{x\} : x \text{ is in the closure of } A\}.$$

Then  $(X, \tau)$  is said to have *countable strong fan tightness at  $x$*  if the selection principle  $S_1(\Omega_x, \Omega_x)$  holds.

In [3] it was shown that for certain “nice” spaces  $(X, \tau)$  it is true that  $S_1(\Omega_x, \Omega_x)$  holds if, and only if, ONE has now winning strategy in the game  $G_1(\Omega_x, \Omega_x)$ . In [3] also an ad hoc example of the failure of this equivalence was given. In Theorem 11 of [5] the following more extreme example is given, assuming the Continuum Hypothesis (CH):

**Theorem 1 (CH).** *There is a  $T_3$  space  $X$  that has countable strong fan tightness at each  $x \in X$ , yet ONE has a winning strategy in  $G_1(\Omega_x, \Omega_x)$  at each  $x \in X$ .*

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## 1. LEMMA 10

For topological space  $(X, \tau)$  let  $\mathfrak{D}$  denote the set  $\{A \subseteq X : A \text{ dense in } X\}$ . Also, let  $\mathfrak{N}\mathfrak{D}$  denote the set  $\{A \subseteq X : A \text{ is not discrete}\}$ . In [5] the proof of Theorem 1 made use of a lemma regarding the infinite game  $G_1(\mathfrak{D}, \mathfrak{N}\mathfrak{D})$ . This game is a dual version of a game introduced in [1] by Berner and Juhasz. More details about this game and construction of the dual game appear in the sources [1, 4, 5].

The claimed proof of Lemma 10 of [5] is flawed. Here is a correct argument:

**Lemma 2.** *Let  $(X, \tau)$  be a  $T_1$ -space with no isolated points. Assume that player ONE has a winning strategy in the game  $G_1(\mathfrak{D}, \mathfrak{N}\mathfrak{D})$  on  $X$ . Then at each  $x \in X$  ONE has a winning strategy in the game  $G_1(\Omega_x, \Omega_x)$ .*

*Proof.* Let  $\sigma$  be a winning strategy for ONE in the game  $G_1(\mathfrak{D}, \mathfrak{N}\mathfrak{D})$ . For each  $x \in X$ , define a strategy  $\sigma_x$  as follows:

- $\sigma_x(\emptyset) = \sigma(\emptyset) \setminus \{x\}$ , and
- for each finite sequence  $(w_1, \dots, w_n)$  of elements of  $X$ ,  $\sigma_x(w_1, \dots, w_n) = \sigma(w_1, \dots, w_n) \setminus \{x\}$ .

Since  $X$  has no isolated points, for each dense set  $D \subset X$  and each  $x \in X$ , the set  $D \setminus \{x\}$  is dense in  $X$ . Thus,  $\sigma_x$  is a strategy for player ONE in the game  $G_1(\mathfrak{D}, \mathfrak{N}\mathfrak{D})$ . Each  $\sigma_x$  play of the game  $G_1(\mathfrak{D}, \mathfrak{N}\mathfrak{D})$  is a  $\sigma$ -play during which TWO never picked the element  $x$ . Thus,  $\sigma_x$  is also a winning strategy for ONE in  $G_1(\mathfrak{D}, \mathfrak{N}\mathfrak{D})$ .

Now we note that at each  $x \in X$  the strategy  $\sigma_x$  is a winning strategy for ONE in the game  $G_1(\Omega_x, \Omega_x)$ : For consider a  $\sigma_x$ -play

$$O_1, w_1, O_2, w_2, \dots, O_n, w_n, \dots$$

For each  $n$ ,  $O_n$  is a dense set not containing the point  $x$ , and thus is an element of  $\Omega_x$ . The set  $\{w_n : 0 < n < \omega\}$  is a discrete subset of  $X$  and does not contain the point  $x$ . Thus, let  $U$  be a neighborhood of  $x$  meeting the set of moves by TWO in at most one point, say  $w_n$ . Since  $x \neq w_n$  and  $X$  is  $T_1$ , there is a neighborhood  $W$  of  $x$  that does not contain  $w_n$ . But then  $U \cap W$  is a neighborhood of  $x$  disjoint from  $\{w_n : 0 < n < \omega\}$ . It follows that  $\{w_n : 0 < n < \omega\}$  is not an element of  $\Omega_x$ .  $\square$

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## REFERENCES

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